

Locating the Zeros of an Analytic Function

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A numerical technique is presented for locating the zeros of an analytic function in the complex plane. The methods used are not new: the important content of this paper is the development and testing of a method to a point where it may be used with confidence and reliability. An application considered here is the location of the eigenvalues of the Orr-Sommerfeld equation for plane Poiseuille flow in a specified portion of the complex (eigenvalue) plane.

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1. INTRODUCTION

There are many problems for which it is necessary to locate, by numerical computation, the complex zeros of a given analytic function. An important class of problems arises when a system of linear differential equations is solved together with boundary conditions. The eigenvalues for such problems consist of those values of some parameter for which a solution of the homogeneous equations exists, as determined by the zeros of a function of that parameter. Many of the phenomena which are currently under investigation using computation as the major tool are so complicated that there is no general theory regarding the spectra. Thus it is essential to have methods for finding the zeros of complex functions which are both powerful and flexible.

The present paper is concerned mainly with the general problem of locating the zeros of an analytic function in a finite region of the complex plane. As a nontrivial illustration of the use of the method, results are given for a particular case of the Orr-Sommerfeld equation for plane Poiseuille flow. This problem was chosen because it has been used previously as a test problem by Antar [2], whose methods are developed further in this paper. In Antar's paper, some 32 eigenvalues are found to varying degrees of precision. Some of his results are correct to only one or two significant figures while others are extremely accurate: moreover, the procedure used for the numerical quadratures did not always converge satisfactorily. A further problem with Antar's method is that it is necessary to compute the derivative of the function whose zeros are sought, and this more than doubles the difficulty of the calculation.

The methods which are developed here stem from work by Delves and Lyness [6] on the location of the zeros of analytic functions. Abd-Elall, Delves, and Reid [1] have also considered the problem of locating zeros and poles of a function, but that is not pursued here. For the Orr–Sommerfeld equation it is shown that highly accurate results may be obtained for the eigenvalues even when ten eigenvalues are simultaneously computed. The basic principles are simple indeed and have a history of at least a hundred years. Suppose that the function whose zeros are sought is denoted by $f(z)$ and that it is analytic in an open region of the complex plane. If C is a regular curve which lies entirely in this region then the integrals

$$S_n = \frac{1}{2\pi i} \int_C z^n \frac{f'(z)}{f(z)} dz \quad (1.1)$$

contain all of the information necessary for the computation of those zeros which lie inside C . The use of these integrals, numerically evaluated, for the extraction of this information was investigated in some detail by Delves and Lyness. They considered three approaches. The first and most direct approach requires explicit evaluation of both $f(z)$ and its derivative $f'(z)$: a FORTRAN implementation of this method by Botten *et al.* has recently been made available [3]. The third approach involves approximating the derivatives using Taylor series, and is not considered here. The second approach uses integration by parts to remove the formal dependence on $f'(z)$ and this leads to multivalued logarithm functions which are difficult to deal with. The main contribution of this paper is to develop a simple method of avoiding the computation of the derivative $f'(z)$ and a simple formula for estimating the errors in evaluating the integrals S_n . The theory is outlined in Section 3; however, the error formulae given there suffer from the fact that they involve the unknown zeros of the function $f(z)$. Section 4 is devoted to the practical implementation of the method including a plausible, but not rigorous, derivation of an error estimate for S_n which may be computed at little cost from $f(z)$. Numerical experiments are reported in Sections 5 and 6. In Section 5 a simple function is chosen to test a number of aspects of the method, and in particular the error estimate (4.10). In Section 6 the Orr–Sommerfeld equation is used as an example which is more realistic. Parameters are chosen so that the results may be compared with known results [2, 12].

It should be mentioned that an alternative approach has been given recently in a paper by Carpentier and Dos Santos [4] and some comments will be made below about the relationship between their approach and the present one. As in [2, 4], circular integration contours are used here, since the integrands are analytic so that the trapezoidal approximation converges exponentially fast even after integration by parts. This is an important consideration and it constitutes good reason for avoiding contours with corners, such as would obtain if a circular region were subject to radial bisection or if a system of squares were used to avoid overlap between adjoining search regions.

2. BASIC PROCEDURE

The basic search region used in this paper is a disc, and no loss in generality is caused by restricting attention to a unit disc centered at the origin in the complex plane. Thus consider the problem of locating the zeros of a function $f(z)$ which lie within the unit circle. It is assumed that $f(z)$ is analytic in some larger disc $|z| < R$, $R > 1$, and that none of the zeros of $f(z)$ lie on the circle $|z| = 1$. In the unit disc $f(z)$ has N zeros at the points a_j , $j = 1, 2, \dots, N$ and there are a further N' zeros in the region $1 < |z| < R$ at the points b_j , $j = 1, 2, \dots, N'$. The latter zeros are important in discussing the numerical implementation of the method. The integrals S_n of Eq. (1.1) may be evaluated by residues to give simple symmetric functions of the interior zeros, viz.,

$$S_n = \sum_{j=1}^N a_j^n. \quad (2.1)$$

This formula is correct even if some of the roots are repeated, that is, a root of order k is counted k times in the sum.

The integrals (1.1) may also be evaluated numerically by methods which are the subject of Sections 3 and 4. Assume that computed values of N and S_n are available. Then it is possible to construct a polynomial $P_N(z)$ whose zeros are the interior zeros a_j . Defining $P_N(z)$ as

$$\begin{aligned} P_N[z] &= \prod_{j=1}^N (z - a_j) \\ &= \sum_{j=0}^N A_j z^{N-j} \end{aligned} \quad (2.2)$$

it is seen that the coefficients A_n are symmetric functions of the roots a_j , albeit different symmetric functions from the integrals S_n . There is a standard relationship which enables the A_n to be computed recursively from S_n . It is [7]

$$\begin{aligned} S_1 - A_1 &= 0 \\ S_2 - A_1 S_1 - 2A_2 &= 0 \\ &\vdots \\ S_k - A_1 S_{k-1} + A_2 S_{k-2} - \dots + (-1)^k k A_k &= 0, \quad k = 1, 2, \dots, N. \end{aligned} \quad (2.3)$$

The problem of finding the roots a_j has now been reduced to that of finding N zeros of an N th order polynomial. Moreover, it is known at the outset that these N roots all lie in the unit disc. Locating these zeros is a simple process. In this work the following procedure is used. First, a zero a_1 is found using Newton's method

with a starting point near the origin. The contribution of this zero is then subtracted from the integrals S_n to define a new set of coefficients $S_n^{(1)}$ by

$$S_n^{(1)} = S_n - a_1^n, \quad n = 0, 1, \dots, N-1, \quad (2.4)$$

and from it, a new polynomial $P_{N-1}(z)$ is constructed using equations (2.3). A zero of this new polynomial is then found and refined using iteration on the original polynomial. If this new zero is closer to the origin than the first one, it replaces it and the procedure is repeated. If not, new coefficients $S_n^{(2)}$ are constructed by the formula

$$S_n^{(2)} = S_n - \sum_{j=1}^2 a_j^n, \quad n = 0, 1, \dots, N-2, \quad (2.5)$$

and from it a new polynomial $P_{N-2}(z)$. A further zero is now found and refined, and this process repeated until N zeros are found, with the restriction that as each new zero is found, any previous zeros further from the origin than it are discarded. In this way, numerical stability of the process of deflating the original polynomial is ensured [5] because the zeros are eliminated in order of increasing magnitude. This method has been found to be completely reliable in practice.

3. NUMERICAL QUADRATURE

In order to implement the method of the last section it is necessary to evaluate the number of zeros, N , and the integrals S_n . In practice, the algorithms needed for this are interdependent, but for convenience of presentation a treatment of the quadrature method is given in this section under the assumption that N is already known.

The aim is to approximate the integrals (1.1) by a quadrature rule, and to eliminate explicit reference to the derivative $f'(z)$. One possibility, employed by Carpentier and Dos Santos [4], is to apply the trapezoidal rule directly to (1.1) and then use integration by parts to eliminate the combination $f'(z)/f(z)$ from the quadrature formulae in favor of a logarithm. The approach here is to apply integration by parts first and then numerical quadrature, which leads to an essentially different set of formulae. Integration by parts will introduce the logarithm function and with it a branch point at each of the zeros of $f(z)$. Since there are N zeros of $f(z)$ inside the contour, the imaginary part of $\ln f$ increases by $2\pi N$ around the contour. However, the logarithm of the function $g(z) = z^{-N}f(z)$ is single valued around the contour. The difference between the analytic properties of $f(z)$ and $g(z)$ is that the latter has an N th order pole at the origin. For $n > 0$, if $g(z)$ is substituted into Eq. (1.1) and residue theory applied, the result is unchanged from when $f(z)$ is used because the pole is at the origin. (Of course, for $n = 0$ the pole does make its

presence felt and the integral changes to zero.) Integration by parts is now possible without any concern about the definition of the branch of the logarithm: this gives

$$S_n = \frac{-n}{2\pi i} \int_C z^{n-1} \ln [z^{-N} f(z)] dz, \quad n > 0. \quad (3.1)$$

For the purpose of numerical quadrature, it is convenient to use polar coordinates,

$$z = e^{i\theta} \quad (3.2)$$

so that the basic formula for S_n becomes

$$S_n = \frac{-n}{2\pi} \int_0^{2\pi} e^{in\theta} [\rho(\theta) + i\sigma(\theta)] d\theta \quad (3.3)$$

where the real functions $\rho(\theta)$ and $\sigma(\theta)$ are defined by

$$\ln[e^{-iN\theta} f(e^{i\theta})] = \rho(\theta) + i\sigma(\theta). \quad (3.4)$$

The crucial point is that the integrand is a periodic function of θ which ensures that the numerical quadrature scheme converges exponentially fast.

The integral is now approximated by the m -point trapezoidal rule, viz.,

$$T_n^m = \frac{-n}{m} \sum_{k=1}^m e^{2\pi i k n/m} [\rho(2\pi k/m) + i\sigma(2\pi k/m)]. \quad (3.5)$$

It is well known [8, 11] that for contour integrals of analytic functions the trapezoidal rule converges exponentially as a function of m . The main purpose of this section is to develop a formula for the error in a form which will give a criterion for deciding the value of m in any particular computation. To this end it is convenient to write

$$\ln[z^{-N} f(z)] = \sum_{j=1}^N \ln(1 - a_j z^{-1}) + \sum_{j=1}^{N'} \ln(b_j - z) + \Psi(z). \quad (3.6)$$

Because $f(z)$ has no singularities in the disc $|z| < R$, the function $\Psi(z)$ is analytic in this region. The integrand in (3.4) is obtained by setting $z = \exp(i\theta)$, so that Eq. (3.5) may be written as

$$\begin{aligned} T_n^m &= \frac{-n}{m} \sum_{j=1}^N \sum_{k=1}^m e^{2\pi i k n/m} \ln(1 - a_j e^{-2\pi i k/m}) \\ &\quad - \frac{n}{m} \sum_{j=1}^{N'} \sum_{k=1}^m e^{2\pi i k n/m} \ln(b_j - e^{2\pi i k/m}) \\ &\quad - \frac{n}{m} \sum_{k=1}^m e^{2\pi i k n/m} \Psi(\exp(2\pi i k/m)). \end{aligned} \quad (3.7)$$

It is convenient to break up the integrals S_n and their approximations T_n^m into three parts as indicated by the three separate sums in this equation. For the integrals, the three parts are

$$\begin{aligned} S_n^{(1)} &= \sum_{j=1}^N a_j^n \\ S_n^{(2)} &= 0 \\ S_n^{(3)} &= 0. \end{aligned} \tag{3.8}$$

The corresponding parts of T_n^m and the errors, are discussed in turn.

The first sum, approximating $S_n^{(1)}$, may be rearranged as

$$\begin{aligned} T_n^{m(1)} &= \frac{n}{m} \sum_{j=1}^N \sum_{k=1}^m e^{2\pi i n k / m} \sum_{l=1}^{\infty} \frac{a_j^l e^{-2\pi i l k / m}}{l} \\ &= \sum_{j=1}^N \sum_{l=1}^{\infty} a_j^l \frac{n}{l m} \sum_{k=1}^m e^{2\pi i (n-l) k / m} \\ &= \sum_{j=1}^N a_j^n \left\{ 1 + \frac{n}{n+m} a_j^m + \frac{n}{n+2m} a_j^{2m} + \dots \right\} \end{aligned} \tag{3.9}$$

which exhibits the error in a useful form. The second sum may be treated similarly to give

$$\begin{aligned} T_n^{m(2)} &= \frac{-n}{m} \sum_{j=1}^{N'} \sum_{k=1}^m e^{2\pi i n k / m} \left\{ \ln b_j - \sum_{l=1}^{\infty} \frac{e^{2\pi i l k / m}}{l b_j^l} \right\} \\ &= \sum_{j=1}^{N'} \sum_{l=1}^{\infty} \frac{n}{l m b_j^l} \sum_{k=1}^m e^{2\pi i (n+l) k / m} \\ &= \sum_{j=1}^{N'} b_j^n \left\{ \frac{n b_j^{-m}}{(m-n)} + \frac{n b_j^{-2m}}{(2m-n)} + \dots \right\} \quad (m > n). \end{aligned} \tag{3.10}$$

The third sum has been treated in detail in Lyness and Delves [11] who show that the following bound may be written

$$\begin{aligned} |T_n^{m(3)}| &\leq \frac{J(R) R^{1-m}}{1 - R^{-m}} \\ J^2(R) &= \frac{1}{2\pi i} \int_{|z|=R} |\Psi(z)|^2 \frac{dz}{z}. \end{aligned} \tag{3.11}$$

From these results it is seen that, for large m , the dominant errors are due to the

zeros closest to the integration contour. An interior zero contributes an error approximated by

$$T_n^{m(1)} - S_n^{(1)} \simeq \frac{na_j^{n+m}}{n+m}, \quad (3.12)$$

while an exterior zero contributes approximately

$$T_n^{m(2)} \simeq \frac{nb_j^{n-m}}{m-n}. \quad (3.13)$$

4. IMPLEMENTATION AND ERROR CONTROL

It is now possible to consider the numerical implementation of the method. The first quantity which is needed in N , the number of zeros of f . It is appropriate here to stress the fact that no simple algorithm will be completely failsafe. Since the evaluation of the integrals S_n for $n > 0$ uses the trapezoidal rule, it is convenient to use equally spaced values of $f(z)$ for later use in the quadrature. Suppose then that values of $f(z)$ are known at a number of equally spaced points around the unit circle. N may now be determined by the principle of argument, since the total increase during one complete circuit is $2\pi N$. However, the argument of $f(z)$ at each point is only determined to within an arbitrary multiple of 2π . To reduce the ambiguity, the argument is arbitrarily chosen to lie in the interval $(-\pi, \pi]$ at some initial point on the circle and then at each successive point around the circle it is chosen so that the jump from point to point is less than π . The maximum jump in the argument will be denoted Φ_{\max} . For functions which are analytic in a disc $|z| < R$, $R > 1$, Φ_{\max} can always be made arbitrarily small by using a sufficiently large number of points. Carpentier and Dos Santos [4] have given a criterion to decide when N is reliably determined from the numerically computed argument based on two tests; viz.,

$$\Phi_{\max} < \Phi_0 \quad (4.1)$$

with $\Phi_0 = 3\pi/4$ and also

$$1/6.1 < |f(z_k)/f(z_{k+1})| < 6.1, \quad k = 1, \dots, m. \quad (4.2)$$

They state that these tests are expected to be sufficient even when the function has double zeros, except in very anomalous cases.

Information about the accuracy of the other integrals S_n , $n > 0$, may also be obtained from Φ_{\max} , using geometric arguments similar to those of [4]. If there is an interior zero a_D which is the main contributor to the jump Φ_{\max} , then reference to Fig. 1 shows that its magnitude $|a_D|$ may be estimated from

$$\tan(\beta/2) \simeq \frac{\sin(\pi/m)}{\cos(\pi/m) - |a_D|} \quad (4.3)$$

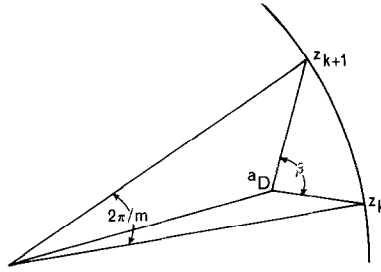


FIG. 1. Geometry of error estimates.

where the equality holds if the argument is mid-way between $\arg(z_k)$ and $\arg(z_{k+1})$. If the zero is of order l , then

$$\Phi_{\max} \simeq l\beta \quad (4.4)$$

so that $|a_D|$ is given as

$$\begin{aligned} |a_D| &\simeq 1 - (\pi/m) \cot(\Phi_{\max}/2l) \\ &\leq \exp \left[-\frac{\pi}{m} \cot \left(\frac{\Phi_{\max}}{2l} \right) \right] \\ &< \exp \left[-\frac{\pi l}{m} \cot \left(\frac{\Phi_{\max}}{2} \right) \right]. \end{aligned} \quad (4.5)$$

When this is used in Eq. (3.12), it gives the following estimate for the contribution to the error from this dominant zero:

$$|T_n^{m(1)} - S_n^{(1)}| \simeq \frac{n}{n+m} \exp[-\pi l \cot(\Phi_{\max}/2)]. \quad (4.7)$$

Similarly, if the dominant zero is just outside the unit circle, at b_D , then Eq. (4.5) becomes

$$|b_D| \simeq 1 + \frac{\pi}{m} \cot \left(\frac{\phi_{\max}}{2l} \right) \quad (4.8)$$

and substitution into Eq. (3.12) gives

$$|T_n^{m(2)}| \simeq \frac{m}{m-n} \exp[-\pi l \cot(\Phi_{\max}/2)] \quad (m > n). \quad (4.9)$$

These results are all that is needed to give a simple criterion to estimate the errors in the integrals from a readily available byproduct of the computation. Since the

total increase in phase is $2\pi N$, Φ_{\max} must exceed $2\pi N/m$ and using this in either (4.7) or (4.9) gives for the dominant contribution to the error from a simple zero the estimate

$$|T_n^m - S_n| \simeq \frac{n}{N} \frac{\Phi_{\max}}{2\pi} \exp \left[-\pi \cot \left(\frac{\Phi_{\max}}{2} \right) \right] \quad (m \gg n). \quad (4.10)$$

The worst case is for $n=N$, and this does not increase with N . It should be emphasised that this is an estimate based on a series of approximations, and in particular on the assumption that the function is not too anomalous. Because of the exponential dependence of the error estimate on Φ_{\max} , and thus on m in the asymptotic (large m) region, it is only the zeros closest to the contour which contribute appreciably to the errors. Thus the estimate of the dominant error is also an estimate of the total error: this has certainly proved to be the case in the numerical experiments to be reported below.

The estimate (4.10) may be used in a practical implementation as follows: First a value of Φ_0 is chosen in (4.1) on the basis of the accuracy required for the evaluation of the integrals S_n . Typically this will be in the range $\pi/3$ to $\pi/5$; in any event it should not exceed $3\pi/4$. The value of $f(z)$ is now computed at eight equally spaced points around the circle, and if either of the tests (4.1) or (4.2) is failed, the number of points is doubled by filling in the midpoints with the new values of $f(z)$ followed by a completely new evaluation of the argument around the new set of points. This process is continued until the criteria are all met or until the maximum allowed value to m is exceeded. In practice the criterion (4.1) becomes the important one once Φ_0 is reduced to values in the suggested range, and we have experienced no problems with an implementation which ignores (4.2) altogether (see next section). As the computation proceeds, a check should also be kept on the expected number of zeros, N . If this number is too large (see next section) then much of the accuracy in the determination of the S_n will be lost in the ensuing determination of the zeros, and the search should be terminated in favor of two or more searches using smaller overlapping circles. Once the zeros have been determined, then they may be refined, for example, by using the secant method or by a repeated use of the present method.

5. SIMPLE TEST PROBLEMS

In this section the method is used on simple functions with known zeros. For the integrals S_n and the zeros a_j , the errors are calculated and compared with the theoretical estimates given in Section 4.

The function used is simply

$$f(z) = \sin(5\pi z - i\pi/2) \cos(5\pi z + i\pi/2) \quad (5.1)$$

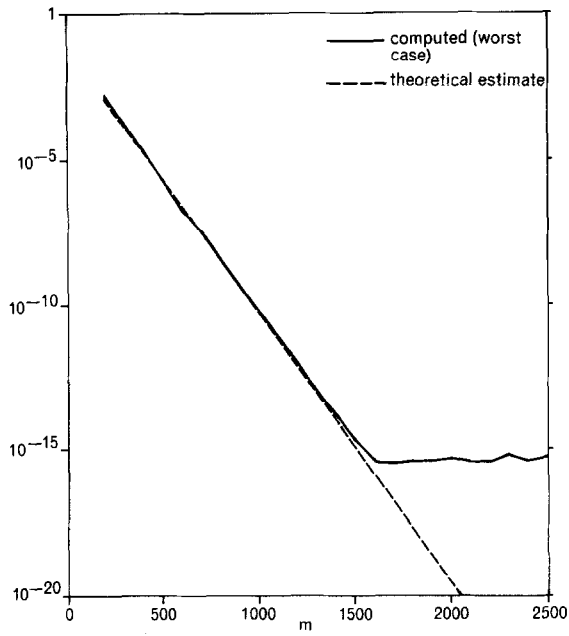


FIG. 2. Errors in S_n as function of m .

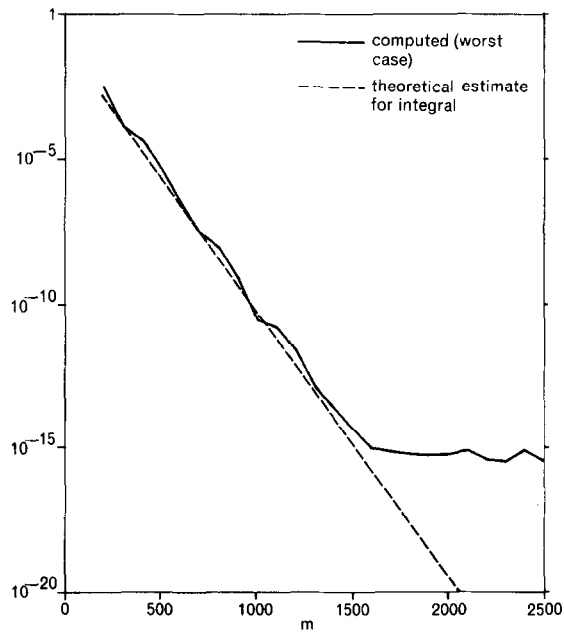


FIG. 3. Errors in zeros as function of m .

which has two infinite rows of zeros on the lines $\text{Im}(z) = \pm i/10$ at

$$\pm \frac{n}{5} + \frac{i}{10}, \quad \pm \frac{2n+1}{10} - \frac{i}{10}, \quad n = 0, 1, 2, \dots$$

For the first test the circle $|z| = \frac{1}{2}$ was used. There are nine zeros inside this circle; the integrals S_1 to S_9 and the corresponding values of these zeros, have been computed for values of m ranging from 200 to 2500 in steps of 100. The calculations were carried out on a Univac 1100/82 using double precision complex arithmetic. The dependence of the errors on m are indicated in Figs. 2 and 3. In presenting the data of Figs. 2 and 3, only the worst error is shown for each value of m . Also shown on each of these figures is the error estimate (4.10). Two facts are evident from these diagrams: first, convergence is at an exponential rate, and second, Eq. (4.10) is a quite reliable criterion for use in an automated search routine.

In a practical implementation of the method as an automated search routine, m will be chosen by requiring Φ_{\max} to be reduced to below some pre-set tolerance while the number of roots will be initially unknown. Thus a second series of tests were carried out using the function (5.1). For these computations, m was adjusted to give $\Phi_{\max} = \pi/9$ while the radius of the circle was increased from 0.5 to 1.5 in steps of 0.1. Each step introduces a further four zeros, so that the largest circle encloses 29 zeros. These tests generate a great deal of data: however, the relevant question relates to the loss of accuracy as the number of zeros increases. It was observed that the least accurate roots are always those closest to the center of the circle (as might be expected). Figure 4 shows the salient trends, plotted as a function of N . For each N , the worst case error in an integral is presented, together with the error estimate (4.10). It is seen that there is no loss of accuracy in the integrals as N increases. Also shown is the worst case error in a root, which is always the error in the root at $z = i/10$, and this is seen to increase steadily as N is

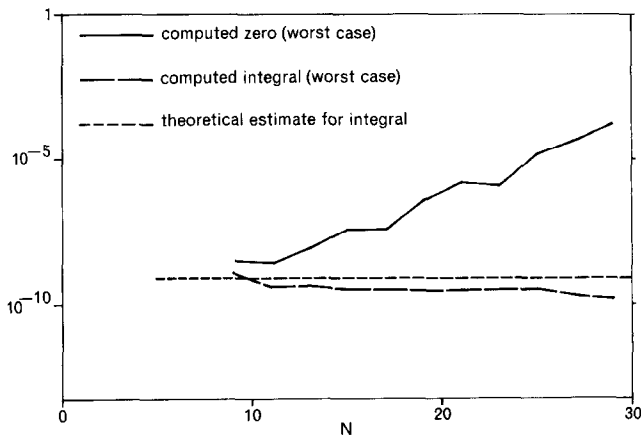


FIG. 4. Errors as function of N .

increased. The conclusion is that up to about 10 zeros may be computed in any one circle to an accuracy which is roughly comparable with that of the integrals. Beyond that, the loss of accuracy in determining the zeros from the integrals becomes increasingly marked, particularly for those zeros which are deepest inside the circle. Nevertheless, large numbers of zeros may be located with considerable precision; certainly with sufficient precision to enable the search to be refined as necessary.

One further set of computations was carried out, in which the radius of the circle was again increased by steps of 0.1 while m was selected in each case to achieve the maximum possible accuracy of the integrals using complex double precision. It was observed that the errors become unmanageable beyond about forty zeros, since at this point all of the precision in the integrals was lost in the algorithms needed to locate the zeros. In fact, any automatic implementation of this method ought to have a maximum value of N set beyond which the calculation is abandoned and a smaller circle selected. Depending on the choice of Φ_{\max} , this maximum could be in the order of 10 to 20.

6. THE ORR-SOMMERFELD PROBLEM

The Orr-Sommerfeld equation describes the stability of two-dimensional, incompressible parallel flow with regard to infinitesimal disturbances [10]. It takes the form

$$(D^2 - \alpha^2)^2 \phi = i\alpha R \{ (U - c)(D^2 - \alpha^2) \phi - \phi D^2 U \} \quad (6.1)$$

where D denotes differentiation with respect to x , α is the disturbance wave number, c its speed, and R is the flow Reynolds number. For the purpose of this paper, the function $U(x)$, which is the undisturbed laminar flow profile, has been taken as $1 - x^2$ which is appropriate for plane Poiseuille flow between parallel plates at $x = \pm 1$. The usual boundary conditions are

$$\begin{aligned} \phi(\pm 1) &= 0 \\ D\phi(\pm 1) &= 0. \end{aligned} \quad (6.2)$$

Mack has shown [12] that for the parameter values $\alpha = 1$, $R = 10,000$, there are 32 eigenvalues in the rectangle $0 \leq \text{Re}(c) \leq 1.0$, $-1.1 \leq \text{Im}(c) \leq 0$. These eigenvalues are given, to five decimal places, in Mack's paper. These were the eigenvalues on which Antar [2] chose to test his method, and they are used again in this paper by locating the zeros of a matching determinant as a function of c . Because of the relatively high value of the Reynolds number, the calculation of the matching determinant to high accuracy is very time consuming and so the tolerance set of this part of the computation was 5 significant figures in the initial search. The search was performed in two stages. First, a number of overlapping circles were chosen with a

radius of 0.15. This value was chosen to limit the number of eigenvalues in any one circle to about ten because the accuracy of $f(c)$ was only 5 significant figures. For each circle, m was increased by powers of two until Φ_{\max} fell below $\pi/6$, at which point the error estimate equation (4.10) shows that further increase in m is pointless without more accurate values of $f(c)$. Errors varied from 2 in the fourth decimal place to complete agreement with Mack's results. The worst cases, as might be expected, were for zeros near the center of the two circles which contain the most zeros. The total number of evaluations of $f(c)$ to this point was 1600, or 50 per zero located. In order to improve the accuracy using the same circles, the number of function evaluations would need to be doubled at least and the accuracy of the evaluations would need to be improved. This is not a sensible strategy when so much information is available at this point. Thus, the second stage of the search consisted of simply using the 32 eigenvalues as the center of 32 new circles, each of radius 0.01, which is about one quarter of the distance between the closest eigenvalues. The actual values of the 32 eigenvalues are not listed here because they are of secondary importance. The results were in complete agreement with Mack for 29 of the 32 eigenvalues. Seventeen of Mack's eigenvalues are taken from Orszag [13], and the two labelled P9 and P10 by Mack have suffered a transposition error. For these two, the present calculation agrees with Orszag rather than Mack. The remaining disagreement is in the real part of the mode labelled S1 by Mack. The real part is given there as 0.67759 whereas the present result is 0.67764. No definitive explanation can be given for this discrepancy, except to emphasize that, in these calculations, it is the Wronskian which is calculated to 7 decimal digits, not just each step in the solution of the differential equation. Thus, the most likely conclusion is that there are some small typographical errors in [12].

It will be observed that the choice of search circles used for this particular numerical experiment used the information on the location of the eigenvalues from Mack's paper. Of course, the method does not depend on this advance knowledge for its success; it has been used here because it is typical of realistic problems that information will be available at an early stage about the qualitative nature of solutions, often from preliminary computation.

7. CONCLUSION

The methods set out in this paper constitute a reliable way of locating the zeros of an analytic function, and of maintaining effective control on the accuracy. Apart from the numerical results reported in Sections 5 and 6, the method is being employed in extensive calculations of the spectra of waves in non-ideal plasmas. No problems have been encountered with the method in this application. On those occasions when an unwise choice is made for a search circle, the progressive numerical results indicate at a fairly early stage of the calculation that a new choice is needed. On no occasion has the method appeared to converge to numbers which later proved to be in error.

In conclusion, a warning must be sounded. The methods of this paper depend in an essential way on the assumed properties of the function $f(z)$. In particular, it is relatively easy in many eigenvalue problems to introduce poles into the determinant or other functions whose zeros are sought, and if this is done, the computations will probably converge quite happily to erroneous results. Any such errors should be revealed when refinement and checking is undertaken, and for this reason alone such practice is recommended. For the same reason, the fact that circles overlap when they are used to search a larger region is an advantage, since it often happens that the same zero is located in more than one circle, and then values may be compared as a check.

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